

FACIAL STRUCTURES OF LATTICE PATH MATROID POLYTOPES

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ABSTRACT. A lattice path matroid is a transversal matroid corresponding to a pair of lattice paths on the plane. A matroid base polytope is the polytope whose vertices are the incidence vectors of the bases of the given matroid. In this paper, we study facial structures of matroid base polytopes corresponding to lattice path matroids.

1. INTRODUCTION

For a matroid M on a ground set $[n] := \{1, 2, \dots, n\}$ with a set of bases $\mathcal{B}(M)$, a *matroid base polytope* $\mathcal{P}(M) := \mathcal{P}(\mathcal{B}(M))$ of M is the polytope in \mathbb{R}^n whose vertices are the incidence vectors of the bases of M . The polytope $\mathcal{P}(M)$ is a face of a *matroid independence polytope* first studied by Edmonds [7], whose vertices are the incidence vectors of all the independent sets in M . There are a lot of research activities about matroid base polytopes for the last few years since it has various applications in algebraic geometry, combinatorial optimization, Coxeter group theory, and tropical geometry. In general, matroid base polytopes are not well understood.

A *lattice path matroid* is a transversal matroid corresponding to a pair of lattice paths having common end points. Many interesting and striking properties of these matroids have been studied. The combinatorial and structural properties of lattice path matroids are given by Bonin et al. in [4] and [5]. The h -vectors, Bergman complexes, and Tutte polynomials of lattice path matroids are studied by several authors [6, 12, 16].

In this paper, we study the facial structure of a *lattice path matroid polytope* which is a matroid base polytope corresponding to a lattice path matroid. This class of matroid base polytopes is belong to important classes of polytopes such as positroid polytopes and generalized permutohedra. Positroid polytopes are studied by Ardila et al. [1] and generalized permutohedra are studied by Postnikov and other authors [13, 14, 15]. Bidkhori [3] provides a description of facets of a lattice path matroid polytope and we extend it to all the faces.

Key words and phrases. matroid base polytope, lattice path matroid.

This paper is organized as follows. In Section 2, definitions and properties of lattice path matroids are given. In Section 3, we define lattice path matroid polytopes and give known results about them. In Section 4, lattice path matroid polytopes for the case of border strips are studied. In particular, we show that all the faces of a lattice path matroid polytope in this case can be described by certain subsets of deletions, contractions and direct sums and express them in terms of a lattice path obtained from the border strip. Section 5 explains facial structures of a lattice path matroid polytope for a general case in terms of certain tilings of skew shapes inside the given region.

2. LATTICE PATH MATROIDS

In this section, we provide basic definitions and properties of lattice path matroids.

A *matroid* M is a pair $(E(M), \mathcal{B}(M))$ consisting of a finite set $E(M)$ and a collection $\mathcal{B}(M)$ of subsets of $E(M)$ that satisfy the following conditions:

- (1) $\mathcal{B}(M) \neq \emptyset$, and
- (2) for each pair of distinct sets B, B' in $\mathcal{B}(M)$ and for each element $x \in B - B'$, there is an element $y \in B' - B$ such that $(B - \{x\}) \cup \{y\} \in \mathcal{B}(M)$.

The set $E(M)$ is called the *ground set* of M and the sets in $\mathcal{B}(M)$ are called the *bases* of M . Subsets of bases are called the *independent sets* of M . It is easy to see that all the bases of M have the same cardinality, called the *rank* of M .

A set system $\mathcal{A} = \{A_j : j \in J\}$ is a multiset of subsets of a finite set S . A *transversal* of \mathcal{A} is a set $\{x_j : x_j \in A_j \text{ for all } j \in J\}$ of $|J|$ distinct elements. A *partial transversal* of \mathcal{A} is a transversal of a set system of the form $\{A_k : k \in K\}$ with $K \subseteq J$.

Edmonds and Fulkerson [8] show the following fundamental result:

Theorem 1. *The transversals of a set system $\mathcal{A} = \{A_j : j \in J\}$ form the bases of a matroid on S .*

A *transversal matroid* is a matroid whose bases are the transversals of some set system $\mathcal{A} = \{A_j : j \in J\}$. The set system \mathcal{A} is called the *presentation* of the transversal matroid. The independent sets of a transversal matroid are the partial transversals of \mathcal{A} .

We consider lattice paths in the plane using steps $E = (1, 0)$ and $N = (0, 1)$. The letters are abbreviations of East and North. We will often treat lattice paths as words in the alphabets E and N , and we will use the notation α^n to denote the concatenation of n letters of α . The *length* of a lattice path $P = p_1 p_2 \cdots p_n$ is n , the number of steps in P .

Definition 2. Let $P = p_1 p_2 \cdots p_{m+r}$ and $Q = q_1 q_2 \cdots q_{m+r}$ be two lattice paths from $(0, 0)$ to (m, r) with P never going above Q . Let $\{p_{u_1}, p_{u_2}, \dots, p_{u_r}\}$ be the set of North steps of P , with $u_1 < u_2 < \cdots < u_r$; similarly, let $\{q_{l_1}, q_{l_2}, \dots, q_{l_r}\}$ be the set of North steps of Q with $l_1 < l_2 < \cdots < l_r$. Let N_i be the interval $[l_i, u_i]$ of integers. Let $M(P, Q)$ be the transversal matroid that has ground set $[m+r]$ and presentation $\{N_i : i \in [r]\}$. The pair (P, Q) is a *lattice path presentation* of $M(P, Q)$. A *lattice path matroid* is a matroid M that is isomorphic to $M(P, Q)$ for some such pair of lattice paths P and Q . We will sometimes call a lattice path presentation of M simply a presentation of M when there is no danger of confusion and when doing so avoids awkward repetition.

The fundamental connection between the transversal matroid $M(P, Q)$ and the lattice paths that stay in the region bounded by P and Q is the following theorem of Bonin et al. [4].

Theorem 3. *A subset B of $[m+r]$ with $|B| = r$ is a basis of $M(P, Q)$ if and only if the associated lattice path $P(B)$ stays in the region bounded by P and Q , where $P(B)$ is the path which has its north steps on the set B positions and east steps elsewhere.*

3. LATTICE PATH MATROID POLYTOPES

Let \mathcal{B} be a collection of r -element subsets of $[n]$. For each subset $B = \{b_1, \dots, b_r\}$ of $[n]$, let

$$e_B = e_{b_1} + \cdots + e_{b_r} \in \mathbb{R}^n,$$

where e_i is the i th standard basis vector of \mathbb{R}^n . The collection \mathcal{B} is represented by the convex hull of these points

$$\mathcal{P}(\mathcal{B}) = \text{conv}\{e_B : B \in \mathcal{B}\}.$$

This is a convex polytope of dimension $\leq n-1$ and is a subset of the $(n-1)$ -simplex

$$\Delta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0, \dots, x_n \geq 0, x_1 + \cdots + x_n = r\}.$$

Gelfand, Goresky, MacPherson, and Serganova [10, Thm. 4.1] show the following characterization of matroid base polytopes.

Theorem 4. *The subset \mathcal{B} is the collection of bases of a matroid if and only if every edge of the polytope $\mathcal{P}(\mathcal{B})$ is parallel to a difference $e_\alpha - e_\beta$ of two distinct standard basis vectors.*

By the definition, the vertices of $\mathcal{P}(M)$ represent the bases of M . For two bases B and B' in $\mathcal{B}(M)$, e_B and $e_{B'}$ are connected by an edge if and only if $e_B - e_{B'} = e_\alpha - e_\beta$. Since the latter condition is equivalent to $B - B' = \{\alpha\}$ and

$B' - B = \{\beta\}$, the edges of $\mathcal{P}(M)$ represent the basis exchange axiom. The basis exchange axiom gives the following equivalence relation on the ground set $[n]$ of the matroid M : α and β are *equivalent* if there exist bases B and B' in $\mathcal{B}(M)$ with $\alpha \in B$ and $B' = (B - \{\alpha\}) \cup \{\beta\}$. The equivalence classes are called the *connected components* of M . The matroid M is called *connected* if it has only one connected component. Feichtner and Sturmfels [9, Prop. 2.4] express the dimension of the matroid base polytope $\mathcal{P}(M)$ in terms of the number of connected components of M .

Proposition 5. *Let M be a matroid on $[n]$. The dimension of the matroid base polytope $\mathcal{P}(M)$ equals $n - c(M)$, where $c(M)$ is the number of connected components of M .*

Bonin et al. [4] give the following result explaining the number of connected components of the lattice path matroid.

Proposition 6. *The lattice path matroid is connected if and only if the bounding lattice paths P and Q meet only at $(0, 0)$ and (m, r) .*

Remind that, for a skew shape bounded by lattice paths P and Q (denoted by $[P, Q]$) and a related rank r lattice path matroid $M[P, Q]$ with a set of bases $\mathcal{B}(M[P, Q])$, the lattice path matroid polytope $\mathcal{P}(M[P, Q])$ is the convex hull $\mathcal{P}(\mathcal{B}(M[P, Q])) = \text{conv}\{e_B = e_{b_1} + \cdots + e_{b_r} : B = \{b_1, \dots, b_r\} \in \mathcal{B}\}$ where e_i is the i th standard basis vector of \mathbb{R}^{m+r} . The following result about the dimension of the lattice path matroid polytope immediately follows from these results.

Corollary 7. *For lattice paths P and Q from $(0, 0)$ to (m, r) with P never going above Q , the dimension of the lattice path matroid polytope $\mathcal{P}(M[P, Q])$ is $m + r - k + 1$, where k is the number of intersection points of P and Q .*

4. LATTICE PATH MATROID POLYTOPES FOR BORDER STRIPS

For a matroid $(E(M), \mathcal{B}(M))$ and a subset S of $E(M)$, let $r(S)$ denote the *rank* of S , the size of largest independent subsets of S . Then, the *restriction* $M|_S$ is the matroid on S having the bases $\mathcal{B}(M|_S) = \{B \cap S : B \in \mathcal{B}(M) \text{ and } |B \cap S| = r(S)\}$, and the *contraction* M/S is the matroid on $E(M) - S$ having the bases $\mathcal{B}(M/S) = \{B - S : B \in \mathcal{B}(M) \text{ and } |B \cap S| = r(S)\}$. For two matroids $(E(M_1), \mathcal{B}(M_1))$ and $(E(M_2), \mathcal{B}(M_2))$, their *direct sum* $M_1 \oplus M_2$ is the matroid on $E(M_1) \cup E(M_2)$ having the bases $\mathcal{B}(M_1 \oplus M_2) = \{B_1 \cup B_2 : B_1 \in \mathcal{B}(M_1), B_2 \in \mathcal{B}(M_2)\}$.

Theorem 8 (Bonin and de Mier [5], 2006). *The class of lattice path matroids is closed under restrictions, contractions, and direct sums.*

The connection between constructions of lattice path matroids and skew shapes bounded by lattice paths P and Q is known as follows. First, we label a step of skew shape $[P, Q]$ by $i + j + 1$ if the step begins at (i, j) . Then, a restriction and a contraction of lattice path matroid $M[P, Q]$ correspond to the deletion of the corresponding region in skew shape $[P, Q]$. If the begin point of one skew shape is attached to the end point of the other, and all the steps are relabeled, we get a direct sum of two matroids. Figure 1 shows how a restriction, a contraction, and a direct sum work to $[P, Q]$.

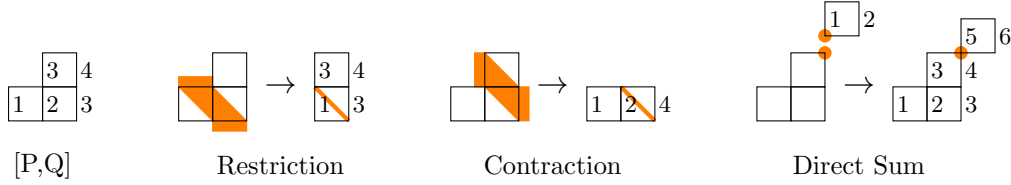


FIGURE 1. Restriction, contraction, and direct sum of matroids on $[P, Q]$.

Our main question is about lattice path matroid polytopes regarding to restrictions, contractions, and direct sums of matroids. Can we use constructions of matroids to figure out the properties of lattice path matroid polytopes? To study facets and describe facial structures of lattice path matroid polytopes, we introduce more specific definitions and notations.

The i -deletion of $M[P, Q]$ is defined by a matroid $M|_{E(M)-\{i\}}$, the restriction of $M[P, Q]$ on $E(M)-\{i\}$. The i -contraction of $M[P, Q]$ is a matroid $M/\{i\} \oplus \{i\}$ which is isomorphic to $M/\{i\}$, the contraction of $M[P, Q]$ on $\{i\}$. An *outside corner* (p, q) of the region $[P, Q]$ is a point on a corner NE on P or a corner EN on Q . At the outside corner (p, q) , let (p, q) -direct sum of $M[P, Q]$ be a matroid $M_1 \oplus M_2$ where a matroid M_1 and M_2 correspond to the lower left quadrant and the upper right quadrant of $[P, Q]$ with the center (p, q) , respectively. If (p, q) is the unique outside corner of $[P, Q]$ such that $p + q = i$ for some integer i , then (p, q) -direct sum is abbreviated to i -direct sum. Figure 2 shows how a deletion, a contraction, and a direct sum work to $[P, Q]$.

Example 9. The polytopes in Figure 3 are facets of the polytope $\mathcal{P}(M[P, Q])$ corresponding to 2-deletion, 4-contraction, and 3-direct sum of the matroid $M[P, Q]$ in Figure 2.

In this section we focus on the properties of lattice path matroid polytopes corresponding to *border strips*, connected (non-empty) skew shapes with no 2×2 square. Let $P = p_1 p_2 \cdots p_{m+r}$ and $Q = q_1 q_2 \cdots q_{m+r}$ be two lattice paths from $(0, 0)$ to (m, r) with P never going above Q . Note that the region

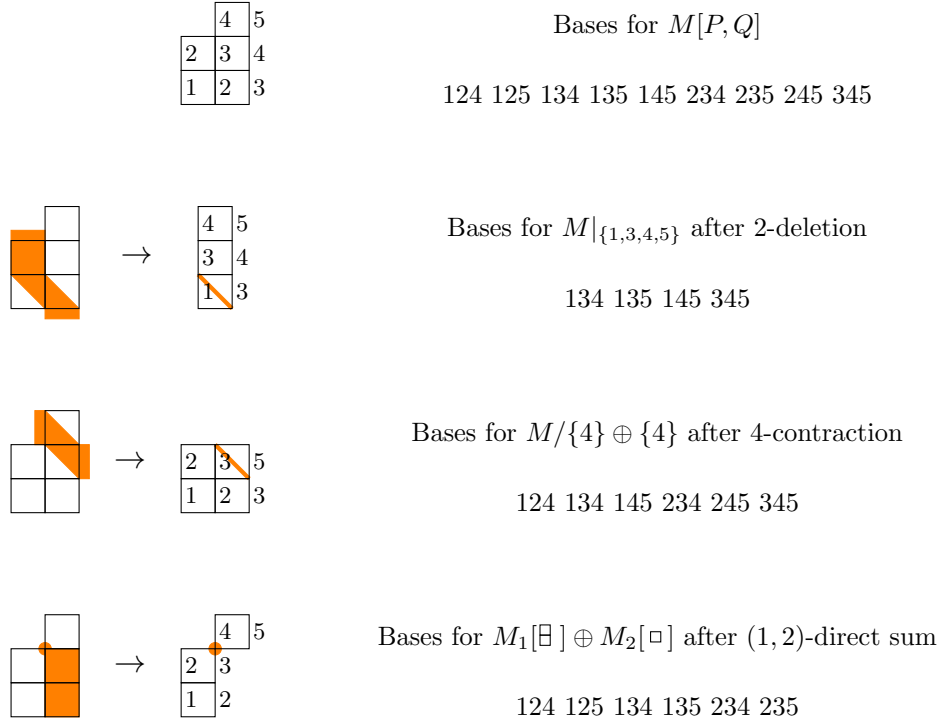


FIGURE 2. 2-deletion, 4-contraction, and (1, 2)-direct sum of matroids on $[P, Q]$ and their bases.

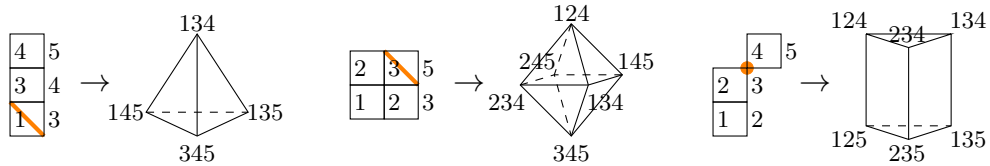


FIGURE 3. 2-deletion, 4-contraction, and 3-direct sum of matroids on $[P, Q]$ and corresponding facets.

bounded by P and Q is a border strip if and only if $p_i = q_i$ for $1 < i < m+r$, $p_1 = q_{m+r} = \text{East step}$, and $q_1 = p_{m+r} = \text{North step}$. For such P and Q where $m+r > 2$, we define a lattice path $R = R(P, Q) = r_1 r_2 \cdots r_{m+r}$ as $r_i = p_i (= q_i)$ for $1 < i < m+r$, $r_1 = r_2$, and $r_{m+r-1} = r_{m+r}$. If $m+r = 2$, that means P and Q are lattice paths from $(0, 0)$ to $(1, 1)$, a lattice path R is defined as two consecutive North steps from $(0, 0)$ to $(0, 2)$. Not only in the case $m+r = 2$, but in general R is not needed to be a path from $(0, 0)$ to (m, r) . See Figure 4.

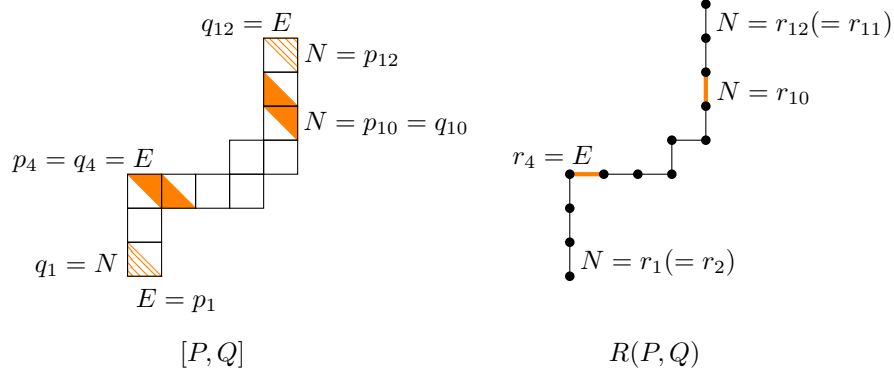


FIGURE 4. Border strip $[P, Q]$ from $(0,0)$ to $(5,7)$ and lattice path $R(P, Q)$ from $(0,0)$ to $(4,8)$.

For a lattice path $R = R(P, Q)$ we define three sets $\mathcal{D}(R)$, $\mathcal{C}(R)$, $\mathcal{S}(R)$ as follows:

$$\begin{aligned}\mathcal{D}(R) &= \{i\text{-deletion of } M[P, Q] : r_i = \text{East step}\}, \\ \mathcal{C}(R) &= \{i\text{-contraction of } M[P, Q] : r_i = \text{North step}\}, \text{ and} \\ \mathcal{S}(R) &= \{i\text{-direct sum of } M[P, Q] : r_i \neq r_{i+1}\}.\end{aligned}$$

For each element of $\mathcal{D}(R(P, Q)) \cup \mathcal{C}(R(P, Q)) \cup \mathcal{S}(R(P, Q))$ we have the corresponding border strip. See Figure 5.

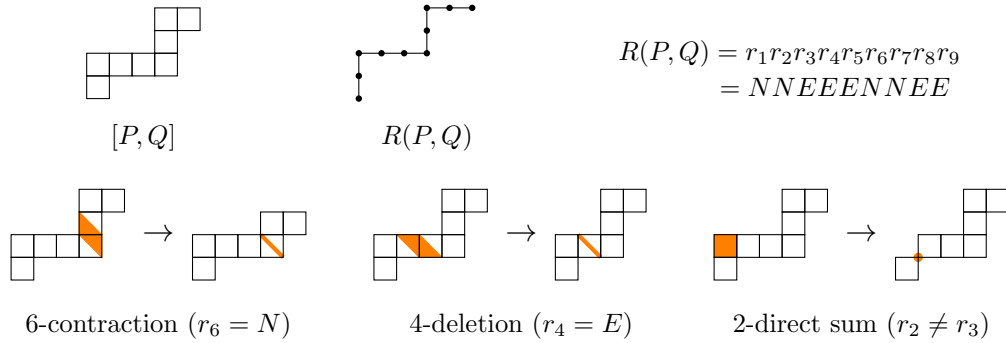


FIGURE 5. Border strips corresponding to deletion, contraction, and direct sum of $M[P, Q]$.

Note that the dimension of $\mathcal{P}(M[P, Q]) = \dim(\mathcal{P}(M[P, Q])) = m + r - 1$ for lattice paths P and Q from $(0,0)$ to (m, r) with P never going above Q where the region bounded by P and Q is a border strip since P and Q satisfy $k = 2$ in Corollary 7.

Lemma 10. *The set of facets of lattice path matroid polytope $\mathcal{P}(M[P, Q])$, where the region bounded by P and Q is a border strip, has a one-to-one correspondence with $\mathcal{D}(R(P, Q)) \cup \mathcal{C}(R(P, Q)) \cup \mathcal{S}(R(P, Q))$.*

Proof. For the polytope $\mathcal{P}(M[P, Q]) = \mathcal{P}(\mathcal{B}(M)) = \text{conv}\{e_B = e_{b_1} + \cdots + e_{b_r} : B \in \mathcal{B}\}$, take a subset $S^{-i} = \text{conv}\{e_B : i \notin B \in \mathcal{B}\}$ on the hyperplane $x_i = 0$ in $\mathbb{R}^{m+r-k+2}$, which is corresponding to i -deletion in $\mathcal{D}(R)$. All the vertices $e_B \in \mathcal{P}(M[P, Q]) - S^{-i}$ lie on the half-space $x_i > 0$ since their i -th coordinates are 1 (> 0). The dimension of S^{-i} is $(m-1) + r - k + 1 = m + r - k$ since k and r are fixed and the only width m dropped by 1 during the i -deletion. Hence, S^{-i} is a facet of $\mathcal{P}(M[P, Q])$.

Similarly for i -contraction in $\mathcal{C}(R)$ if we take a subset $S^{+i} = \text{conv}\{e_B : i \in B \in \mathcal{B}\}$ on the hyperplane $x_i = 1$ in $\mathbb{R}^{m+r-k+2}$, all the vertices of $\mathcal{P}(M[P, Q]) - S^{+i}$ have i -th coordinate 0 (< 1) and lie on the half-space of the hyperplane $x_i < 1$. After the i -contraction the height r dropped by 1, while m and k are fixed, and the dimension of S^{+i} is $m + (r-1) - k + 1 = m + r - k$. Hence, S^{+i} is also a facet of $\mathcal{P}(M[P, Q])$.

For i -direct sum in $\mathcal{D}(R)$, without loss of generality, we may assume that r_i is East step and r_{i+1} is North step. That means a direct sum occurs at the point (p, q) of the path Q where $p + q = i$. Take a subset $S^i = \text{conv}\{e_B : |[i] \cap B| = q, B \in \mathcal{B}\}$ of $\mathcal{P}(M[P, Q])$. Then, S^i lies on the hyperplane $x_1 + x_2 + \cdots + x_i = q$ and the other points on the half-space $x_1 + x_2 + \cdots + x_i < q$. The dimension of S^i is $m + r - (k+1) + 1 = m + r - k$ since we have the same end points and get one more intersection point after i -direct sum. Hence, S^i is a facet of $\mathcal{P}(M[P, Q])$.

For the other direction, to show is all the facets of $\mathcal{P}(M[P, Q])$ are on the hyperplanes corresponding to $\mathcal{D}(R(P, Q)) \cup \mathcal{C}(R(P, Q)) \cup \mathcal{S}(R(P, Q))$. Suppose a polytope $\mathcal{P}(M[P, Q])$ has a facet not lying on the hyperplane from $\mathcal{D}(R(P, Q)) \cup \mathcal{C}(R(P, Q)) \cup \mathcal{S}(R(P, Q))$. That means hyperplanes corresponding to $\mathcal{D}(R(P, Q)) \cup \mathcal{C}(R(P, Q)) \cup \mathcal{S}(R(P, Q))$ do not generate $\mathcal{P}(M[P, Q])$ in its affine hull $x_1 + x_2 + \cdots + x_{m+r-k+2} = r$. Then, there exists a point $x = (x_1, x_2, \dots, x_{m+r-k+2}) \in \mathbb{R}^{m+r-k+2} - \mathcal{P}(M[P, Q])$, located in the intersection of all the half-spaces mentioned in the previous paragraphs and $x_1 + x_2 + \cdots + x_{m+r-k+2} = r$.

Since x satisfies above conditions, for any maximal sequence of consecutive E 's of R , $r_u r_{u+1} \cdots r_{u+v}$, we have $x_i \geq 0$ for $i \in [u, u+v]$ and $x_1 + x_2 + \cdots + x_u \geq N(R, u)$ and $x_1 + x_2 + \cdots + x_{u+v} \leq N(R, u) + 1$ where $N(R, i)$ is the number of North steps until i th step of R . Then, $x_i \leq 1$ for $i \in [u, u+v]$, and this implies $0 \leq x_i \leq 1$ if $r_i = E$. Similarly, for any maximal sequence $r_u r_{u+1} \cdots r_{u+v}$ of consecutive N 's of R , we obtain $0 \leq x_i \leq 1$ for $i \in [u, u+v]$, and this means $0 \leq x_i \leq 1$ if $r_i = N$. Hence, we get $0 \leq x_i \leq 1$ for all i in $[m+r]$.

For each sequence $r_j r_{j+1}$ of R such that $r_j r_{j+1} = NE$ or $j = m + r$, two conditions, $x_j \leq 1$ and $x_1 + x_2 + \cdots + x_j \geq N(R, j) = N(P, j)$, are given. Hence, it follows that $x_1 + x_2 + \cdots + x_{j-1} \geq N(R, j) - 1 = N(R, j - 1) = N(P, j - 1)$. If r_{j-1} is a North step, $x_{j-1} \leq 1$ and $x_1 + x_2 + \cdots + x_{j-2} \geq N(R, j - 1) - 1 = N(R, j - 2) = N(P, j - 2)$. If r_{j-1} is an East step, $x_1 + x_2 + \cdots + x_{j-2} \geq N(R, h) = N(R, j - 2) = N(P, j - 2)$ where $r_h r_{h+1}$ is a previous NE sequence of R or $h = 1$. After checking each East step r_{j-k} for $1 \leq k \leq j - h - 1$, we have $x_1 + x_2 + \cdots + x_i \geq N(P, i)$ for all i in $[m + r]$. If we apply a similar way to each subsequence $r_j r_{j+1}$ of R such that $r_j r_{j+1} = EN$ or $j = 1$ where $x_j \geq 0$ and $x_1 + x_2 + \cdots + x_j \leq N(R, j) + 1 = N(Q, j)$, we also get $x_1 + x_2 + \cdots + x_i \leq N(Q, i)$ for all i in $[m + r]$.

Hence, we conclude $0 \leq x_i \leq 1$ and $N(P, i) \leq x_1 + x_2 + \cdots + x_i \leq N(Q, i)$ for all $i \in [m + r]$. This is a contradiction to the fact that x is not a point in $\mathcal{P}(M[P, Q])$. Therefore, hyperplanes corresponding to $\mathcal{D}(R(P, Q)) \cup \mathcal{C}(R(P, Q)) \cup \mathcal{S}(R(P, Q))$ generate $\mathcal{P}(M[P, Q])$ in its affine hull $x_1 + x_2 + \cdots + x_{m+r-k+2} = r$, and all the facets of $\mathcal{P}(M[P, Q])$ are on the hyperplanes corresponding to $\mathcal{D}(R(P, Q)) \cup \mathcal{C}(R(P, Q)) \cup \mathcal{S}(R(P, Q))$. \square

Corollary 11. *For lattice paths P and Q from $(0, 0)$ to (m, r) with the region bounded by P and Q as a border strip, the number of facets of $\mathcal{P}(M[P, Q])$ is $m + r + d$ where d is the number of outside corners of the region $[P, Q]$.*

Not only facets of lattice path matroid polytope, we will find a one to one corresponding set for all the faces of lattice path matroid polytope.

If we consider a lattice path $R(P, Q)$ as a sequence on $\{E, N\}^{m+r}$ and cut the sequence R at every direct sum position, we may get $d + 1$ subsequences where $d = |\mathcal{S}(R(P, Q))|$ = the number of corners of $R(P, Q)$. Let $(S_1, S_2, \dots, S_{d+1})$ be a set partition of $\mathcal{D}(R(P, Q)) \cup \mathcal{C}(R(P, Q))$ with S_i corresponding to i th subsequence of R . Define a set $S_i^L = S_i \cup \{(i-1)\text{th direct sum}\}$ for $1 < i \leq d+1$ and $S_1^L = S_1$. Similarly, define a set $S_i^R = S_i \cup \{i\text{th direct sum}\}$ for $1 \leq i < d+1$ and $S_{d+1}^R = S_{d+1}$.

For a subset T of $\mathcal{D}(R(P, Q)) \cup \mathcal{C}(R(P, Q)) \cup \mathcal{S}(R(P, Q))$ we introduce the following three conditions:

- (C1) $S_i^R \not\subseteq T$ for $2 \leq i \leq d + 1$.
- (C2) For each sequence $S_i^L, S_{i+1}, \dots, S_d$ where $1 \leq i \leq d$, a subsequence $S_i^L, S_{i+1}, \dots, S_j$ can be included in T if and only if the subsequence has an even-length and $S_{j+1} \not\subseteq T$.
- (C3) A sequence $S_i^L, S_{i+1}, \dots, S_{d+1} - \{(m+r)\text{-deletion}, (m+r)\text{-contraction}\}$ can be included in T for $1 \leq i \leq d$ if and only if the sequence has an even-length.

Example 12. For a lattice path $R(P, Q) = E^2 N^2 E N E^3 N E N^4$, we have $d = 7$ and $S_1 = \{1\text{-deletion}, 2\text{-deletion}\}$, $S_2 = \{3\text{-contraction}, 4\text{-contraction}\}$,

$S_3 = \{5\text{-deletion}\}, \dots, S_8 = \{12\text{-contraction}, \dots, 15\text{-contraction}\}$. Figure 6 shows condition (C1), (C2), and (C3).

(a) The orange arrows represent 7 sets, $S_2^R, S_3^R, \dots, S_7^R$, and $S_8^R (= S_8)$, which are not included in T by (C1). As an example, T cannot be a 4-subset such as $\{3\text{-contraction}, 4\text{-contraction}, 4\text{-direct sum}, 5\text{-deletion}\}$ since it contains S_2^R .

(b) The orange arrow is a sequence S_2^L, S_3, \dots, S_7 . By condition (C2), $\{2\text{-direct sum}, 3\text{-contraction}, 4\text{-contraction}\}$ cannot be T since it contains S_2^L , but not S_3 . However, $\{2\text{-direct sum}, 3\text{-contraction}, 4\text{-contraction}, 5\text{-deletion}\}$ can be T since it includes S_2^L and S_3 , but not S_4 .

(c) As we see the green and blue arrows in Figure 6(b), we need the third condition (C3) if $j = d+1$. The green arrow represents the last condition that T can include a sequence $S_3^L, S_4, S_5, S_6, S_7, S_8 - \{15\text{-contraction}\}$, and the last condition for the blue arrow is that a sequence $S_1^L, S_2, S_3, S_4, S_5, S_6, S_7, S_8 - \{15\text{-contraction}\}$ can be contained in T .

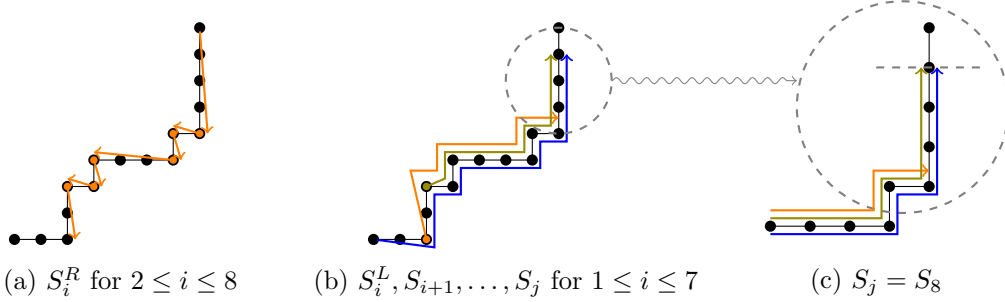


FIGURE 6. Corners on $R(P, Q)$.

Theorem 13. *For lattice paths P and Q from $(0, 0)$ to (m, r) with P never going above Q and the region bounded by P and Q being a border strip, the set of $(m + r - 1 - t)$ -dimensional faces of the lattice path matroid polytope $\mathcal{P}(M[P, Q])$ has a one-to-one correspondence with t -subsets of $\mathcal{D}(R(P, Q)) \cup \mathcal{C}(R(P, Q)) \cup \mathcal{S}(R(P, Q))$ satisfying condition (C1), (C2), and (C3).*

Proof. Note that $(m + r - 1 - t)$ -dimensional faces of lattice path matroid polytope $\mathcal{P}(M[P, Q])$ are facets of $(m + r - t)$ -dimensional faces of $\mathcal{P}(M[P, Q])$. First, to show is each construction, in t -subsets of $\mathcal{D}(R(P, Q)) \cup \mathcal{C}(R(P, Q)) \cup \mathcal{S}(R(P, Q))$ satisfying condition (C1), (C2), and (C3), is possible after other constructions have been done while the dimensions are being dropped from $m + r - 1$ to $m + r - 1 - t$. That means, any s structures from t -subsets drop the dimension of $\mathcal{P}(M[P, Q])$ by s exactly, where $1 \leq s \leq t$. It is not hard to check using the similar steps in Lemma 10.

For the other direction, suppose that $S_i^R \subseteq T$ for some i ($2 \leq i \leq d+1$) in (C1). Since i -direct sum drops more than 1 dimension of the polytope obtained by $|S_i|$ contractions (or deletions) in S_i^R , the dimension of faces after applying all the constructions in S_i^R is less than $m+r-1-|S_i^R|$. Similarly, suppose an odd length sequence $S_i^L, S_{i+1}, \dots, S_j \subseteq T$ and $S_{j+1} \not\subseteq T$ for some i and j such that $1 \leq i < j \leq d$ in (C2). Then, the dimension of faces after all the constructions in $S_i^L, S_{i+1}, \dots, S_j \subseteq T$ are applied is less than $m+r-1-(|S_i^L|+|S_{i+1}|+\dots+|S_j|)$ since $(i-1)$ -direct sum drops more than 1 dimension of the polytope obtained after $|S_i|+|S_{i+1}|+\dots+|S_j|$ contractions and deletions. As the same way, $(i-1)$ -direct sum also drops more than 1 dimension of the polytope in (C3). Therefore, the set of $(m+r-1-t)$ -dimensional faces of lattice path matroid polytope $\mathcal{P}(M[P, Q])$ has a one-to-one correspondence with t -subsets of $\mathcal{D}(R(P, Q)) \cup \mathcal{C}(R(P, Q)) \cup \mathcal{S}(R(P, Q))$ satisfying condition (C1), (C2), and (C3). \square

5. GENERAL CASE

In this section, we consider the region $[P, Q]$ as a connected (non-empty) skew shape, and the faces of a lattice path matroid polytope $\mathcal{P}(M[P, Q])$ are described in terms of certain tiled subregions without 2×2 rectangles inside $[P, Q]$. Note that, even if the subregion is not allowed, $[P, Q]$ may contain 2×2 rectangles unlike previous sections.

We begin with the following proposition which is a generalization of Lemma 10. Since $[P, Q]$ is a connected skew shape now, P and Q have only two intersection points $(0, 0)$ and (m, r) , and R from $[P, Q]$ is a sequence corresponding to a border strip from $(0, 0)$ to (m, r) contained in $[P, Q]$. Its proof is omitted since it is similar to the proof of Lemma 10.

Proposition 14. *The set of facets of lattice path matroid polytope $\mathcal{P}(M[P, Q])$ has one-to-one correspondence with the disjoint union of the following three sets:*

$$\begin{aligned} \mathcal{D}(P, Q) &= \{i\text{-deletion} : r_i = \text{East step for some } R \text{ from } [P, Q]\}, \\ \mathcal{C}(P, Q) &= \{i\text{-contraction} : r_i = \text{North step for some } R \text{ from } [P, Q]\}, \text{ and} \\ \mathcal{S}(P, Q) &= \{(p, q)\text{-direct sum} : (p, q) \text{ is an outside corner of } [P, Q]\}. \end{aligned}$$

Note that if the region $[P, Q]$ is a border strip, the above three sets coincide with $\mathcal{D}(R), \mathcal{C}(R)$, and $\mathcal{S}(R)$ in the previous section, respectively.

Since a face of $\mathcal{P}(M[P, Q])$ is a facet of a one-higher-dimensional face of $\mathcal{P}(M[P, Q])$, Proposition 14 implies that all the faces of $\mathcal{P}(M[P, Q])$ are obtained after applying deletions, contractions, and direct sums. Note that a set of matroid constructions used to generate a face of $\mathcal{P}(M[P, Q])$ may not be uniquely determined.

Before we give a description of faces of $\mathcal{P}(M[P, Q])$, we need to define several notions. We label each unit box having points (i, j) and $(i + 1, j + 1)$ inside the region $[P, Q]$ with $i + j + 1$, and let (i, j) be the *starting point* and $(i + 1, j + 1)$ be the *ending point* of the box. A *block* is a border strip located inside $[P, Q]$, and we may consider a block as a tableau with labels of boxes in the block. The starting point and ending point of a block are the starting point of the smallest labelled box and the ending point of the largest labelled box contained in the block respectively. The *clones* inside the region $[P, Q]$ are blocks such that they are the same as tableaux and distinguishable only by the difference in their positions. A block is a clone of itself. In Figure 7, the block with the starting point $(1, 0)$ and the ending point $(3, 2)$ is a clone of the block with the starting point $(0, 1)$ and the ending point $(2, 3)$, and vice versa.

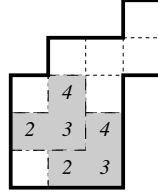


FIGURE 7. Clones labeled by 2-3-4 in $[P, Q]$

For some subregion $[\lambda, \mu]$ of the region $[P, Q]$ and some tiling τ of $[\lambda, \mu]$, where λ and μ are lattice paths from $(0, 0)$ to (m, r) in $[P, Q]$ with λ never going above μ , we define a *block-tiled region* (abbreviated BTR) $[\lambda, \mu]_\tau$ as follows:

- (1) Each maximal continuous intersection of λ and μ passes through an outside corner or an end point of $[P, Q]$.
- (2) τ uses blocks as tiles. That means the set of all the unit boxes in $[\lambda, \mu]$ is covered by blocks without gaps or overlaps.
- (3) If two blocks have a same-labeled-box in $[\lambda, \mu]_\tau$, they are clones to each other.

Note that a block-tiled region is not always defined for any subregion or any tilling. For two block-tiled regions $[\lambda, \mu]_\tau$ and $[\lambda', \mu']_{\tau'}$, we say $[\lambda, \mu]_\tau$ covers $[\lambda', \mu']_{\tau'}$ if $[\lambda, \mu]_\tau$ can be obtained by attaching a clone of a block in $[\lambda', \mu']_{\tau'}$ below λ' or above μ' . If a block-tiled region is not covered by any other, it is called a *maximal block-tiled region*.

Lemma 15. *There is a one-to-one correspondence between the set of all the faces of $\mathcal{P}(M[P, Q])$ and the set of all the maximal block-tiled regions inside $[P, Q]$.*

Proof. For a face σ of $\mathcal{P}(M[P, Q])$, take a set of matroid constructions by which σ is obtained from $\mathcal{P}(M[P, Q])$. If (p, q) -direct sum is in the set where (p, q) is an outside corner of P or Q , we remove all the steps inside $[P, Q]$ which lie strictly northwest or southeast of the point (p, q) respectively. Also, if i -contraction or i -deletion is in the set, all the East steps or North steps with the label i inside $[P, Q]$ are removed respectively. After removing all the steps corresponding to constructions in the set, if we delete remaining steps not on the connected lattice paths from $(0, 0)$ to (m, r) lastly, we end up with a maximal block-tiled region $[\lambda, \mu]_\tau$ inside $[P, Q]$. Conversely, if some operations in $\mathcal{D}(P, Q) \cup \mathcal{C}(P, Q) \cup \mathcal{S}(P, Q)$, which should be used to get the maximal block-tiled region $[\lambda, \mu]_\tau$ from $[P, Q]$, are used to the polytope $\mathcal{P}(M[P, Q])$, one can easily check that the obtained corresponding face is the face σ . \square

A *block-tiled band* is a block-tiled region containing no 2×2 squares. We say block-tiled bands inside $[P, Q]$ are in the same *family* if the sets of maximal continuous intersections and clones constituting them are the same. A *block-tiled bottom* is a block-tiled band which is the lowest one among block-tiled bands in its family. Note that a block-tiled band bordered by λ inside a maximal block-tiled region $[\lambda, \mu]_\tau$ is a block-tiled bottom. We say that two blocks are *adjacent* if they share a step on their boundaries. Note that, if two adjacent blocks in a block-tiled region $[\lambda, \mu]_\tau$ share two or more steps, they are clones each other. A block in a block-tiled band can be adjacent at most two other blocks.

There is a one-to-one correspondence between the set of all the maximal block-tiled regions inside $[P, Q]$ and the set of all the block-tiled bottoms inside $[P, Q]$. For a given maximal block-tiled region $[\lambda, \mu]_\tau$ inside $[P, Q]$, one can remove all the clones except the lowest ones to get a block-tiled bottom $[\lambda, \nu]_\tau$ inside $[P, Q]$ where we identify λ with the lower bounding path of the Young diagram of $[\lambda, \mu] (= \lambda \setminus \mu)$, and μ and ν with upper bounding paths of the Young diagrams of $[\lambda, \mu]$ and $[\lambda, \nu]$ respectively. The inverse is obtained by inserting all the clones of each block in the block-tiled bottom $[\lambda, \nu]_\tau$ into a region $[\nu, \mu]$ where μ is the upper bounding path of the highest block-tiled band among the family members of $[\lambda, \nu]_\tau$. See Example 16. Hence, there is a one-to-one correspondence between the set of all the faces of $\mathcal{P}(M[P, Q])$ and the set of all the block-tiled bottoms inside $[P, Q]$ by Lemma 15.

Example 16. Let $P = E^3 N^3 E N^2$ and $Q = N^3 E N E^2 N E$. The shaded block-tiled region shown in Figure 8(a) is a block-tiled bottom inside $[P, Q]$. Since the block-tiled region in Figure 8(b) is obtained by inserting the clone of the block labeled by 2-3-4 in the shaded block-tiled bottom, it covers the block-tiled region in Figure 8(a). Hence, the block-tiled region in Figure 8(a) is not maximal. If we also insert the clone of the single block labeled by 5 as in

Figure 8(c), the upper bound of the region in Figure 8(c) is not a lattice path. Hence, the region in Figure 8(c) is not a skew shape, and not a block-tiled region. Therefore, the striped block-tiled band in Figure 8(b) is the highest family member of the shaded block-tiled bottom, and the blocked-tiled region in Figure 8(b) is the maximal block-tiled region corresponding to the shaded block-tiled bottom.

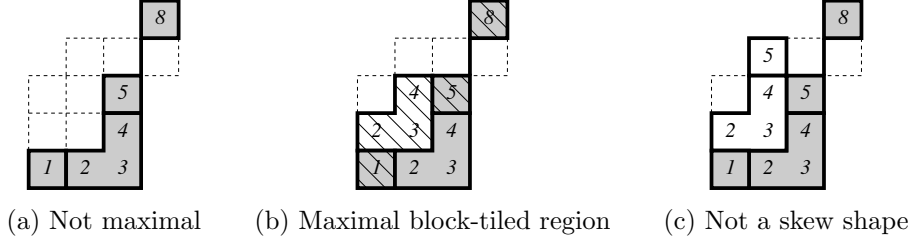


FIGURE 8. Correspondence between block-tiled bottoms and maximal block-tiled regions

The next proposition describes the covering relation in the face poset of $\mathcal{P}(M[P, Q])$ in terms of block-tiled bottoms. The proof is straightforward and is omitted.

Proposition 17. *Codimension 1 subfaces of an n -dimensional face corresponding to a block-tiled bottom $[\lambda, \nu]_\tau$ inside $[P, Q]$ are obtained as follows:*

- (1) *(Direct sum at an outside corner) For an outside corner (p, q) of $[P, Q]$, let $[P, Q]_{(p, q)}$ be the subregion of $[P, Q]$ which can be obtained after (p, q) -direct sum works to $[P, Q]$. If there exists a family member f of $[\lambda, \nu]_\tau$ such that $f \setminus [P, Q]_{(p, q)}$ consists of a single block containing a starting point $(p, *)$, an ending point $(*, q)$, and an outside corner (p, q) , then one can take the lowest one among such f 's and remove the single block keeping the lattice path through $(p, *)$, (p, q) , and $(*, q)$. See Example 18.*
- (2) *(Deletion of a block) Let i be the smallest box label of a block B_1 in $[\lambda, \nu]_\tau$. If B_1 is adjacent to only one block B_2 whose labels are bigger than those of B_1 , delete B_1 from the family members of $[\lambda, \nu]_\tau$ keeping the perimeter of B_1 from $(0, 0)$ to the starting point of the clone of B_2 so that new block-tiled bands with one less blocks than $[\lambda, \nu]_\tau$ are obtained. Note that the new bands fall into at most two types of families. One can get the block-tiled bottoms corresponding to $(n - 1)$ -dimensional subfaces by taking the lowest ones by families. If the perimeter begins with an East step (a North step), the obtained bottom corresponds to the i -deletion (respectively, i -contraction). See*

Figure 10(c) and 10(d) in Example 19. If B_1 is adjacent to only one block whose labels are smaller than those of B_1 , similar operations can be done.

If B_1 is not adjacent to any other block, one can either replace λ by removing all the boxes of B_1 , or replace ν by adding all the boxes of B_1 . This corresponds to the i -contraction or the i -deletion, respectively. See Figure 10(e) and 10(f) in Example 19.

- (3) (Merge of adjacent blocks) If two blocks B_1 and B_2 in $[\lambda, \nu]_\tau$ are adjacent along a step in this order, one can merge B_1 and the clone of B_2 in the family members of $[\lambda, \nu]_\tau$ by deleting the step between B_1 and the clone of B_2 so that new block-tiled bands with one less blocks than $[\lambda, \nu]_\tau$ are gained. Note that the new bands also fall into at most two types of families. One can obtain the block-tiled bottoms corresponding to $(n-1)$ -dimensional subfaces by taking the lowest ones by families. If the deleted step is a North step (an East step), this corresponds to the i -deletion (respectively, i -contraction). See Figure 10(g) and 10(h) in Example 19.

Example 18. For the region $[P, Q]$ where $P = E^5 N^2 E^2 N E^2 N^4$ and $Q = N^4 E^4 N^3 E^5$, one can have an outside corner $(4, 4)$ of Q and a block-tiled bottom as in Figure 9(a). Note that the maximal block-tiled region corresponding to the block-tiled bottom in Figure 9(b) consists of all the family members of the given bottom. The shaded block-tiled band shown in Figure 9(c) is the lowest family member satisfying conditions in case (1) of Proposition 17. One can obtain the new block-tiled bottom from the lowest family member after removing the 8-labeled block in Figure 9(d).

Example 19. Let $P = E^2 N E^3 N^4 E N$ and $Q = N^3 E N^2 E^2 N E^3$. Figure 10(a) shows the maximal block-tiled region for a 7-dimensional face of $\mathcal{P}(M[P, Q])$. The maximal block-tiled region for the face obtained from $(5, 5)$ -direct sum from this face is shown in Figure 10(b). Figures 10(c)-(h) show codimension 1 faces of the 6-dimensional face corresponding to the maximal block-tiled region shown in Figure 10(b). They are examples for cases (2) and (3) of Proposition 17.

The following corollary follows from Proposition 17.

Corollary 20. *All the n -dimensional faces of the polytope $\mathcal{P}(M[P, Q])$ are corresponding to the block-tiled bottoms with n blocks inside the region $[P, Q]$.*

Proof. By Proposition 7, the dimension of a lattice path matroid polytope $\mathcal{P}(M[P, Q])$ having $m+r-1$ single-boxed-blocks in the block-tiled bottom of $[P, Q]$ is $m+r-1$. The result follows since each covering relation in the face

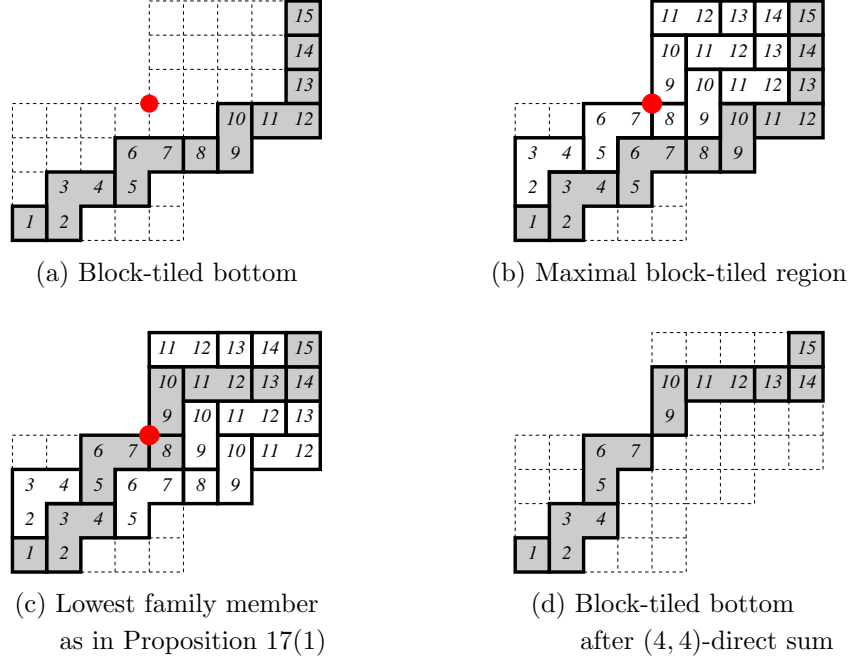
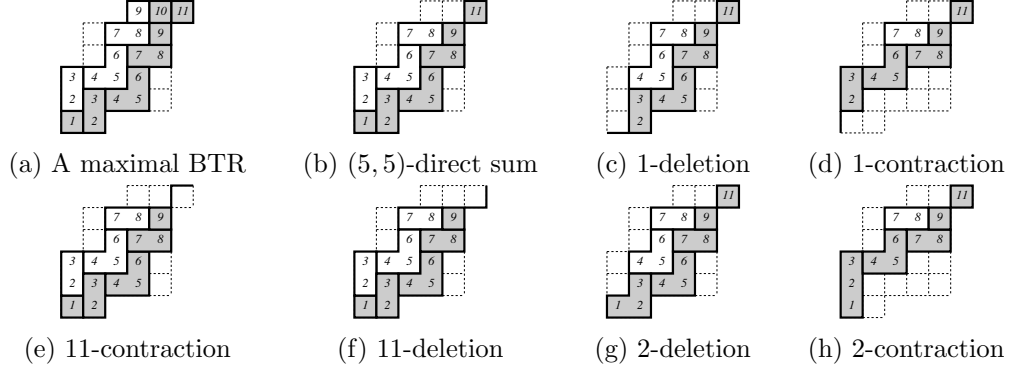
FIGURE 9. Block-tiled bottoms corresponding to faces of $\mathcal{P}(M[P, Q])$ 

FIGURE 10. Direct sum/Deletion/Contraction

poset of $\mathcal{P}(M[P, Q])$ reduces the number of blocks in the block-tiled bottom by one. \square

In the following proposition, we give a more explicit proof for a special case of Corollary 20.

Proposition 21. *All the edges of the polytope $\mathcal{P}(M[P, Q])$ are corresponding to the block-tiled bottoms with 1 block inside the region $[P, Q]$.*

Proof. Take an edge $e_B e_{B'}$ of the polytope $\mathcal{P}(M[P, Q])$ where

$$e_B = e_{b_1} + \cdots + e_{b_r} = (x_1, x_2, \dots, x_n) \in \Delta_n \text{ and}$$

$$e_{B'} = e_{b'_1} + \cdots + e_{b'_r} = (y_1, y_2, \dots, y_n) \in \Delta_n$$

for $(n-1)$ -simplex $\Delta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0, \dots, x_n \geq 0, x_1 + \cdots + x_n = r\}$. By Theorem 4, without loss of generality, there exist j and k ($1 \leq j < k \leq n$) such that

$$\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $x_i = y_i$ ($1 \leq i \leq n$) for the other coordinates.

Because the associated lattice path $P(B)$ is identical with $P(B')$ before the j th step and $\begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we have a starting point of a region R_1 at the j th step. Similarly, we have an ending point of a region R_2 at the k th step. Since $P(B)$ and $P(B')$ also have identical sequences between j th step and k th step after being separated only by 1 step at j th step, $P(B)$ and $P(B')$ do not intersect between j th step and k th step, and 2×2 squares cannot be contained between $P(B)$ and $P(B')$. Hence, R_1 and R_2 are the same skew-shaped region without 2×2 squares. Therefore, there is a unique block generated by $P(B)$ and $P(B')$ inside $[P, Q]$. \square

Consider the lattice path matroid polytope $\mathcal{P}(M[P, Q])$ and a lattice path L inside the region $[P, Q]$. We define $L' = L'(P, L)$ as the lattice path such that L' passes all the intersection points of P and L and, for each maximal connected region between P and L with the starting point (x, y) and the ending point $(x+a, y+b)$, L' has a sequence $E^a N^b$ from (x, y) to $(x+a, y+b)$.

Corollary 22. *For the lattice path matroid polytope $\mathcal{P}(M[P, Q])$, the number of edges of this polytope is equal to the sum of the areas between L and L' where the sum is over all lattice paths L from $(0, 0)$ to (m, r) inside the region $[P, Q]$.*

Proof. For a lattice path L from $(0, 0)$ to (m, r) inside the region $[P, Q]$ and a unit box U with the starting point (u_1, u_2) inside the region $[L', L]$, one can construct the block-tiled bottom $[\lambda, L]_\tau$ inside $[P, Q]$ such that $[\lambda, L]_\tau$ has only one block with the starting point $(*, u_2)$ and the ending point $(u_1 + 1, *)$ on L , and λ is identical with L before $(*, u_2)$ and after $(u_1 + 1, *)$. If a block-tiled bottom $[\lambda, \nu]_\tau$ with 1 block inside $[P, Q]$ is given, one can take a lattice path ν and a unit box with the starting point $(u_1 - 1, u_2)$ where u_1 is the x -coordinate of the ending point and u_2 is the y -coordinate of the starting point of the block.

Hence, there is a one to one correspondence between block-tiled bottoms with 1 block inside $[P, Q]$ and the pairs of a lattice path L from $(0, 0)$ to (m, r)

inside $[P, Q]$ and a unit box inside $[L', L]$. By Proposition 21, the number of edges of the polytope $\mathcal{P}(M[P, Q])$ is equal to the sum of areas between L and L' with summation over all lattice paths L from $(0, 0)$ to (m, r) inside the region $[P, Q]$. \square

Corollary 22 is a nice generalization of the following result [3, Lemma 3.6]. Note that $L' = P$ in the case P is the lattice path $E^m N^r$.

Corollary 23. *Consider the lattice path matroid polytope $\mathcal{P}(M[E^m N^r, Q])$ where Q is a lattice path from $(0, 0)$ to (m, r) . The number of edges of this polytope is equal to the sum of areas below the path L with summation over all lattice paths L from $(0, 0)$ to (m, r) inside the region $[E^m N^r, Q]$.*

6. FUTURE WORKS

The **cd**-index $\Psi(\mathcal{P})$ of a polytope \mathcal{P} , a polynomial in the noncommutative variables **c** and **d**, is a very compact encoding of the flag numbers of a polytope \mathcal{P} [2]. The third author shows how the **cd**-index of a polytope can be expressed when a polytope is cut by a hyperplane [11]. For lattice path matroids of rank 2, the following is obtained from [11].

Proposition 24. *For lattice paths $P = E^{\alpha+\beta} N E^\gamma N$ and $Q = N E^\alpha N E^{\beta+\gamma}$ with $\alpha+\beta+\gamma = m$, the **cd**-index of the lattice path matroid polytope $\mathcal{P}(M[P, Q])$ is*

$$\begin{aligned} \Psi(\mathcal{P}(M[P, Q])) &= \sum_{i=\alpha+1}^{\alpha+\beta} \Psi(\mathcal{P}(M_i)) - \left(\sum_{i=\alpha+2}^{\alpha+\beta} \Psi(\Delta_i \times \Delta_{m-i+2}) \right) \cdot \mathbf{c} \\ &\quad - \sum_{(0,0) < (i,j) \leq (\alpha,\gamma)} \binom{\alpha+1}{i} \binom{\gamma+1}{j} \left(\sum_{k=2}^{\beta-2} \Psi(\Delta_{\alpha-i+k} \times \Delta_{\beta-j+\gamma-k+2}) \right) \cdot \mathbf{d} \cdot \Psi(\Delta_{i+j}), \end{aligned}$$

where $M_i = M[E^i N E^{m-i} N, N E^{i-1} N E^{m-i+1}]$ and Δ_j is $(j-1)$ -dimensional simplex.

It is known that a hyperplane split of a lattice path matroid polytope $\mathcal{P}(M[P, Q])$ can occur when $[P, Q]$ contains a 2×2 square [3]. Using descriptions of faces of a lattice path matroid polytope appeared in Sections 4 and 5, we would like to understand the **cd**-index of a lattice path matroid polytope of rank greater than 2.

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